UDC 539.3

BIORTHOGONAL EXPANSIONS IN PROBLEMS OF MECHANICS

PMM Vol. 43, No. 4, 1979, pp. 698-708
G. Ia. POPOV
(Odessa)
(Received June 28, 1978)

The author has shown in [1] how biorthogonal systems of functions can be used to obtain explicit solutions of problems of mechanics (in the form of series in terms of the specified systems) which could not be obtained by other methods. Below it is shown that such solutions can be constructed for problems which can be described by the following equation:

$$Lg = f \quad (f \in H,\ g \in H') \tag{0.1}$$

where L is a linear operator acting from the Hilbert space H' to the Hilbert space H. The plane contact problem of the theory of elasticity is used to illustrate the method.

1. The systems $\{\xi_n\} = \xi_0, \xi_1, \ldots, \xi_n, \ldots; \quad \{\eta_n\} = \eta_0, \eta_1, \ldots, \eta_n, \ldots$ of the elements of H shall be called biorthonormed or B-systems [2] if

$$(\xi_n, \eta_m) = \delta_{nm};\ \delta_{nm} = 0,\ n \neq m;\ \delta_{nn} = 1$$

The B-systems are important, since for any element $f \in H$ we can obtain the following formal expansions:

$$f = \sum_{m=0}^{\infty} (f, \xi_m)\, \eta_m, \quad f = \sum_{m=0}^{\infty} (f, \eta_m)\, \xi_m \tag{1.1}$$

To construct the B-systems we introduce two systems of elements $\{\varphi_n\}$ and $\{\psi_n\}$ and construct the determinants of the type

$$B_n = \begin{vmatrix} b_{00} & b_{01} & \ldots & b_{0n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n-1,0} & b_{n-1,1} & \ldots & b_{n-1,n} \\ b_{n0} & b_{n1} & \ldots & b_{nn} \end{vmatrix} \quad \begin{pmatrix} n = 0, 1, 2, \ldots \\ B_0 = b_{00} \end{pmatrix} \tag{1.2}$$

$$b_{nm} = (\varphi_n, \psi_m) \tag{1.3}$$

We shall call the systems $\{\varphi_n\}$ and $\{\psi_n\}$ B-linearly independent if $B_n \neq 0$ $(n = 0, 1, 2, \ldots)$.

A method of constructing the B-systems in explicit form is given by

Theorem 1. The systems $\{\xi_n\}$ and $\{\eta_n\}$ constructed from any B-linearly independent systems $\{\varphi_n\}$ and $\{\psi_n\}$ according to the formulas

$$\xi_n = \frac{\xi_n^*}{\sqrt{B_n B_{n-1}}}, \quad \xi_n^* = \begin{vmatrix} b_{00} & b_{10} & \ldots & b_{n0} \\ \cdot & \cdot & \cdot & \cdot \\ b_{0,n-1} & b_{1,n-1} & \ldots & b_{n,n-1} \\ \varphi_0 & \varphi_1 & \ldots & \varphi_n \end{vmatrix} \tag{1.4}$$

$$\eta_n = \frac{\eta_n^*}{\sqrt{B_n B_{n-1}}}, \quad \eta_n^* = \begin{vmatrix} b_{00} & b_{01} & \dots b_{0n} \\ \cdot & \cdot & \cdot \\ b_{n-1,0} & b_{n-1,1} & \dots b_{n-1,n} \\ \psi_0 & \psi_1 & \dots \psi_n \end{vmatrix}$$

$$(n = 0, 1, 2, \dots; B_{-1} = 1; \xi_0^* = \varphi_0; \eta_0^* = \psi_0)$$

are B-systems.

The proof of the theorem is simple and follows the known proofs (see [3]). The explicit formulas obtained for the B -systems can be transformed into the following recurrent formulas:

$$\begin{matrix}\xi_n \\ \eta_n\end{matrix} = A_n \left[\begin{matrix}\varphi_n \\ \psi_n\end{matrix} - \sum_{j=0}^{n-1} \begin{matrix}(\varphi_n, \eta_j)\,\xi_j \\ (\psi_n, \xi_j)\,\eta_j\end{matrix} \right] \tag{1.5}$$

$$A_n^{-2} = b_{nn} - \sum_{j=0}^{n-1} (\psi_n, \xi_j)(\varphi_n, \eta_j)$$

In certain particular important cases the formulas (1. 2), (1. 4) can be considerably simplified. Let e. g.

$$(\varphi_n, \psi_m) = b_{nm} = 0, \quad n < m \tag{1.6}$$

Then we obviously have

$$B_n = \prod_{j=0}^{n} b_{jj}, \quad \eta_n^* = B_{n-1}\psi_n$$

Expanding the first determinant in (1. 4) by the first column and using (1. 6), we obtain

$$\xi_n^* = \sum_{m=0}^{n} c_{nm}\varphi_m, \quad c_{nm} = (-1)^{n-m}\Delta_{nm}B_{m-1}$$

Here Δ_{nm} denote the $(n - m)$-th order, almost triangular determinants

$$\Delta_{nm} = \begin{vmatrix} b_{m+1,m} & b_{m+2,m} & \dots b_{n-1,m} & b_{n,m} \\ b_{m+1,m+1} & b_{m+2,m+1} & \dots b_{n-1,m+1} & b_{n,m+1} \\ 0 & b_{m+2,m+2} & \dots b_{n-1,m+2} & b_{n,m+2} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots b_{n-1,n-1} & b_{n,n-1} \end{vmatrix}$$

$$m = 0, 1, 2, \dots, n-1$$

$$\Delta_{nn} = 1, \quad \Delta_{n,n-1} = b_{n,n-1}$$

$$\Delta_{n,n-2} = b_{n,n-1}b_{n-1,n-2} - b_{n-1,n-1}b_{n,n-2}$$

Performing the elementary operations (beginning from the lowest row), we can reduce this determinant to the triangular form and obtain

$$\Delta_{nm} = b_{n,n-1}b^*_{n-1,n-2}b^*_{n-2,n-3}\dots b^*_{m+2,m+1}b^*_{m+1,m} \tag{1.7}$$

$$m = 0, 1, 2, \dots, n-1$$

$$b_{k,j}^* = b_{k,j} - \frac{b_{kk}}{b_{k+1,k}^*} b_{k+1,j}^* \quad \begin{pmatrix} k = n-1, n-2, n-3, \ldots \\ j = 0, 1, 2, \ldots \quad b_{n,j}^* = b_{n,j} \end{pmatrix}$$

2. To solve equation (0.1) we require a system of elements $\{\zeta_n\}$ defined by the explicit

$$\zeta_n = \frac{\zeta_n^*}{\sqrt{B_n B_{n-1}}}, \quad \zeta_n^* = \begin{vmatrix} b_{00} & b_{01} & \ldots b_{0n} \\ \cdot & \cdot & \cdot \\ b_{n-1,0} & b_{n-1,1} & \ldots b_{n-1,n} \\ \chi_0 & \chi_1 & \ldots \chi_n \end{vmatrix} \quad (\zeta_0^* = \chi_0) \tag{2.1}$$

or recurrence

$$\zeta_n = A_n \left[\chi_n - \sum_{j=0}^{n-1} (\psi_n, \xi_j)\, \zeta_j\right] \tag{2.2}$$

formulas, with $\{\chi_n\} \in H'$.

Let us introduce the linear operators A and B acting in H_A and H_B respectively, with the corresponding domains of definition in H and H' respectively. Let us take any two systems $\{\sigma_n\} \in H_A$ and $\{\tau_n\} \in H_B$ and construct the systems $\{\varphi_n\} \in H$ and $\{\psi_n\} \in H'$ using formulas

$$\varphi_n = A\sigma_n, \; \psi_n = LB\tau_n, \; n = 0, 1, 2, \ldots \tag{2.3}$$

Assuming now that the above systems are B-linearly independent, we use them as the initial systems for constructing the B-systems (1.4). After this we construct the system $\{\zeta_n\} \in H'$ using the formula (2.1) and taking into account (2.3) where the system $\{\chi_n\} \in H'$ is defined by the formula

$$\chi_n = B\tau_n, \quad n = 0, 1, 2, \ldots \tag{2.4}$$

We shall show now that the formal solution of (0.1) is given in explicit form by the formula

$$g = \sum_{m=0}^{\infty} (f, \xi_m)\, \zeta_m = \sum_{m=0}^{\infty} (f, \xi_m^*) \frac{\zeta_m^*}{B_m B_{m-1}} \tag{2.5}$$

Indeed, from (2.3) and (2.4) we have $L\chi_n = \psi_n$. Taking this into account and comparing the second formula of (1.4) with (2.1), we find that

$$L\zeta_n = \eta_n \tag{2.6}$$

Taking this into account and operating with L on the series (2.5), we arrive at the first series of (1.1).

Let us note some particular cases of the solution (2.5). Putting e.g. $B = I$, $A = (L^*)^{-1}$ in (2.3) and (2.4) (i.e. A is the converse of the conjugate operator L^*), $\sigma_n = \tau_n = \chi_n$ $(H_A = H_B = H' = H)$ and assuming that the system $\{\chi_n\}$ is complete in H and orthonormal, we find that

$$(A\sigma_n, LB\tau_m) = (L^{*-1}\chi_n, L\chi_m) = (\chi_n, \chi_m) = \delta_{nm}$$

and hence

$$\varphi_n = \xi_n = L^{*-1}\chi_n, \qquad \psi_n = \eta_n = L\chi_n$$

With this in mind, we find that the formula (2.5) is transformed into the known formula (see [4], p. 373)

$$g = \sum_{m=0}^{\infty} (f, L^{*-1}\chi_m)\,\chi_m \tag{2.7}$$

If on the other hand we assume that

$$A = L,\ B = I,\ \sigma_n = \tau_n = \chi_n,\ H_A = H_B = H' \tag{2.8}$$

then from (2.3) we find that $\varphi_n = \psi_n = L\chi_n$, and comparison of the formulas (1.4) and (2.1) yields the relation $L\zeta_n = \xi_n = \eta_n$. The formula (2.5) which represents the solution of (0.1) will in this case assume the form

$$g = \sum_{m=0}^{\infty} (f, L\zeta_m)\,\zeta_m \tag{2.9}$$

which was obtained using a different approach by Kozin [5].

We obtain another important case by setting in (1.2)—(1.4) and (2.1)—(2.4)

$$A = B = I,\quad \sigma_n = \tau_n = \chi_n = \varphi_n,\quad H_A = H_B = H' = H \tag{2.10}$$

and assuming that the operator L is symmetrical [6]. This yields the relations

$$\zeta_n = \xi_n, \quad L\xi_n = \eta_n \tag{2.11}$$

In this case the solution of (0.1) will assume the form

$$g = \sum_{m=0}^{\infty} (f, \xi_m)\,\xi_m \tag{2.12}$$

which coincides with the formula obtained by a different route by Mikhlin in [7].

Finally, setting in (2.1)—(2.4) and (1.2)—(1.4)

$$A = I,\ B = L^*,\ \tau_n = \sigma_n = \varphi_n,\ H_A = H_B = H' = H \tag{2.13}$$

we arrive at the relations

$$\chi_n = L^*\varphi_n,\ \eta_n = LL^*\xi_n,\ L\zeta_n = \eta_n \tag{2.14}$$

and it can be shown that in this case the solution (2.5) of (0.1) represents in explicit form the Enskog [8] method of solving the integral Fredholm equations.

We also note that if

$$(f, \xi_m) = 0, \quad m > N$$

then the series (2.3) is replaced by

$$g = \sum_{m=0}^{N} (f, \xi_m)\,\zeta_m$$

This will occur e.g. in the case when

$$f = \sum_{j=0}^{N} a_j\psi_j \quad (\psi_j = LB\tau_j) \tag{2.15}$$

where a_j are arbitrary constants, since the method of constructing the first system of (1.4) implies that $(\psi_j, \xi_m) = 0$ when $j < m$.

3. We illustrate the application of the above formulas by considering the plane contact problem of impressing a die with symmetric profile $b(x)$ without corners, into an elastic half-space. As we know [9], the problem can be reduced to that of solving the integral equation (3.1) with conditions (3.2)

$$\int_{-a}^{a} \ln \frac{1}{|x-\xi|} p(\xi)\, d\xi = c - b(x), \quad |x| \leqslant a \tag{3.1}$$

$$\int_{-a}^{a} p(x)\, dx = P, \quad p(\pm a) = 0 \tag{3.2}$$

Here $p(x)$ is the contact stress sought, P is a given compressive force, $2a$ denotes the length of the region of contact in question and c is an arbitrary constant (which can be eliminated by setting in (3.1) $x = 0$ and subtracting the resulting expression from (3.1)). As a result we have

$$x = as, \ \xi = a\sigma, \ f(s) = -b(as), \ q(\sigma) = ap(a\sigma) \tag{3.3}$$

and in place of (3.1) we can write

$$Lg = \int_{-1}^{1} \ln \left| \frac{\sigma}{\sigma - s} \right| \sqrt{1-\sigma^2}\, g(\sigma)\, d\sigma = f(s), \quad |s| \leqslant 1 \tag{3.4}$$

Here the unknown function $q(s)$ is replaced by $g(s)$ continuous over the range $-1 \leqslant s \leqslant 1$ and connected with it by the formula

$$q(s) = \sqrt{1-s^2}\, g(s) \tag{3.5}$$

and this enables us to satisfy, a priori, the second condition of (3.2) with the first condition assuming the form

$$\int_{-1}^{1} q(s)\, ds = \int_{-1}^{1} \sqrt{1-s^2}\, g(s)\, ds = P \tag{3.6}$$

For the particular case of (0.1) obtained here, we take $L^2(-1,1)$, as H' and $L_w^2(-1,1)$ as H, with the scalar product of the type

$$(g, f) = \int_{-1}^{1} (1-s^2)^{-1/2} g(s) f(s)\, ds$$

In order to utilize the formula (2.5) in solving the integral equation (3.4), we must fix the elements in (2.3) and (2.4), and we shall assume in these formulas that

$$A = I, \quad H_A = H = L_w^2(-1,1), \quad B = I, \quad H_B = H' = L^2(-1,1) \tag{3.7}$$

$$\tau_n = \chi_n = U_n(s), \quad \varphi_n = \sigma_n = t_n(s) = \begin{cases} -4T_0(s)\pi^{-2}, & n = 0 \\ 4n\pi^{-2}T_n(s), & n = 1, 2, \ldots \end{cases}$$

where $T_n(s)$ and $U_n(s)$ denote the Chebyshev (Tschebyscheff) polynomials of first and second kind. Moreover we have

$$\psi_m = LU_m = \int_{-1}^{1} \ln \left| \frac{\sigma}{\sigma - s} \right| \sqrt{1-\sigma^2}\, U_m(\sigma)\, d\sigma$$

Computing the last integral with help of the formula

$$\int_{-1}^{1} \ln \frac{1}{|\sigma - s|} \frac{U_m(\sigma)\, d\sigma}{(1-\sigma^2)^{-1/2}} = \begin{cases} -\dfrac{(m-1)!\,\pi}{2^{-m-2}\,(2m+1)!!} P_{m+2}^{-3/2,\,-3/2}(s), & m \geqslant 1 \\ {}^1/_4\pi\,[2\ln 2 - T_2(s)], & m = 0 \end{cases}$$

which follows from the results of [10], we obtain

$$\psi_0 = LU_0 = \frac{\pi}{4}[T_2(0) - T_2(s)] = -\frac{\pi s^2}{2} \tag{3.8}$$

$$\psi_m = LU_m = \frac{(m-1)!\,\pi}{2^{-m-2}\,(2m+1)!!}[P_{m+2}^{-3/2,\,-3/2}(0) - P_{m+2}^{-3/2,\,-3/2}(s)], \quad m \geqslant 1$$

The Jacobi polynomials appearing here can be written in terms of the Chebyshev polynomials according to the formula

$$4P_{m+2}^{-3/2,\,-3/2}(s) = \frac{\Gamma({}^3/_2 + m)}{\sqrt{\pi}\,(m+1)!}\left(\frac{m}{m+2}T_{m+2}(s) - T_m(s)\right), \quad m = 0, 1, 2, \ldots$$

by utilizing the known recurrence formulas (see [11]).

Let us now turn to the problem of constructing the B-systems (1.4). We calculate the scalar product

$$(\varphi_n, \psi_m) = \int_{-1}^{1} \frac{t_n(s)}{\sqrt{1-s^2}}\, ds \int_{-1}^{1} \ln\left|\frac{\sigma}{\sigma - s}\right| \sqrt{1-\sigma^2}\, U_m(\sigma)\, d\sigma$$

by changing the order of integration and using the relations [10]

$$\frac{1}{\pi}\int_{-1}^{1} \ln \frac{1}{|s-\sigma|} \frac{T_n(s)}{\sqrt{1-s^2}}\, ds = \begin{cases} \ln 2 \cdot T_0(\sigma), & n = 0 \\ n^{-1}T_n(\sigma), & n = 1, 2, 3, \ldots \end{cases}$$

Next we use the formulas

$$\int_{-1}^{1} \sqrt{1-\sigma^2}\, U_m(\sigma)\, T_n(\sigma)\, d\sigma = \frac{\pi}{4}\delta_{n,\,m} - \frac{\pi}{4}\delta_{n-1,\,m+1}, \quad n \geqslant 1$$

to arrive at

$$(\varphi_n, \psi_m) = (t_n, \psi_m) = \delta_{m,\,n} - \delta_{m+1,\,n-1}, \quad n \geqslant 1 \tag{3.9}$$

$$(\varphi_0, \psi_m) = (t_0, \psi_m) = \lambda_m, \quad m = 0, 1, 2, \ldots$$

$$\lambda_0 = 1, \quad \lambda_m = -\frac{(m-1)!\,2^{m+4}}{(2m+1)!!} P_{m+2}^{-3/2,\,-3/2}(0)$$

which can also be written as

$$\lambda_{2m-1} = 0, \quad \lambda_{2m} = (-1)^m \frac{(2m-1)!\,2^{2m+1}\,({}^3/_4)_m({}^5/_4)_m}{(4m+1)!!(m+1)!({}^1/_2)_m}, \quad m = 1, 2, \ldots \tag{3.10}$$

provided that we use the formula

$$(-1)^m P_{2m}^{\alpha,\alpha}(0) = \frac{(1/2+\alpha/2)_m (1+\alpha/2)_m}{(1+\alpha)_m m!}$$

The latter formula can be obtained using the generating function for the Jacobi polynomilas $P_n^{\alpha,\beta}(x)$ for $\alpha = \beta$ [11].

Let us now compute the determinants (1.2), (1.4) and (2.1). Although in the case in question the condition (1.6) is not completely fulfilled, nevertheless we succeed in completing the computations of the determinants in question using the operations described below. Writing out the determinant (1.2) with (3.9) taken into account and expanding it by the elements of the bottom row, we find that

$$B_{2m+1} = B_{2m}, \quad B_{2m} = \lambda_{2m} + B_{2m-2}, \quad m = 0, 1, 2, \ldots; \quad B_{-2} = 0$$

Consequently, taking (3.10) into account we obtain

$$B_{2m+1} = B_{2m} = \sum_{j=0}^{m} \lambda_{2j} = \tag{3.11}$$

$$1 + \sum_{j=1}^{m} \frac{(-1)^j (2j-1)!\, 2^{2j+1} (3/4)_j (5/4)_j}{(4j+1)!!(j+1)!(1/2)_j}, \quad m = 0, 1, 2, \ldots$$

Expanding the first determinant of (1.4) for $n = 2m$ $(m = 0, 1, 2, \ldots)$ by the elements of the last column and using (3.9), we find that

$$\xi_{2m}^*(s) = B_{2m-2} t_{2m} + \xi_{2m-2}^* = \sum_{j=1}^{m} B_{2j-2} t_{2j}(s) + \xi_0^*, \quad m = 1, 2, \ldots \tag{3.12}$$

$$\xi_0^* = \varphi_0 = t_0(s)$$

For the odd $n = 2m+1$ the determinants in question can be reduced, by virtue of (3.9), to the triangular form (by performing the obvious linear transformations of the columns). The corresponding calculations yield

$$\xi_{2m+1}^*(s) = B_{2m} \sum_{j=0}^{m} t_{2j+1}(s), \quad m = 0, 1, 2, \ldots \tag{3.13}$$

The second determinants of (1.4) with condition (3.9) are also easily calculated for the odd $n = 2m+1$

$$\eta_{2m+1}^*(s) = B_{2m} \psi_{2m+1}(s), \quad m = 0, 1, 2, \ldots \tag{3.14}$$

In the case of $n = 2m$, we should make the last term of the first row equal to zero by linear combination of the first and last row, and reduce the algebraic complement of the last term of the bottom row to the triangular form. This, with (3.11) taken into account, yields the formula

$$\eta_{2m}^*(s) = B_{2m-2}\psi_{2m} - \lambda_{2m} \sum_{j=0}^{m-1} \psi_{2j}, \quad m = 1, 2, \ldots \tag{3.15}$$

$$\eta_0^* = \psi_0(s)$$

It is obvious that the determinants (2.1) can be obtained from the determinants just discussed by replacing ψ_m by $\chi_m = U_m(s)$, i.e.

$$\zeta_{2m}^*(s) = B_{2m-2}U_{2m}(s) - \lambda_{2m}\sum_{j=0}^{m-1} U_{2j}(s) = B_{2m-2}U_{2m}(s) - \tag{3.16}$$

$$\frac{\lambda_{2m}}{2}(1-s^2)^{-1}[1 - T_{2m}(s)], \quad m = 1, 2, \ldots$$

$$\zeta_0^* = \chi_0 = U_0(s); \quad \zeta_{2m+1}^*(s) = B_{2m}U_{2m+1}(s), \quad m = 0, 1, 2, \ldots$$

Substituting the formulas (3.11)—(3.16) into (2.5) enables us to write the explicit expression for the solution of the integral equation (3.4) with (3.5) taken into account

$$q(s) = \sqrt{1-s^2}\sum_{m=0}^{\infty}\int_{-1}^{1}\frac{\xi_m^*(\sigma)f(\sigma)}{\sqrt{1-\sigma^2}}d\sigma\,\frac{\zeta_m^*(s)}{B_m B_{m-1}} \tag{3.17}$$

We note an important (for the contact problems [9]) particular case of (3.4) just solved, in which the right hand side has the form

$$f(s) = \sum_{j=1}^{N} b_j s^{2j}$$

Using (3.8) we can write it, in this case, as a linear combination of the functions $\psi_{2j}(s)$, i.e. we arrive at the case (2.15). In this case the series (3.17) representing the solution of (3.4) becomes a finite sum.

In particular, when a parabolic die acts on an elastic half-space and we have [9]

$$f(s) = -\gamma a^2 s^2 \quad (\gamma = \text{const})$$

then the series (3.17) is reduced to its first term and (3.11)—(3.16) together with (3.7) yield

$$q(s) = \frac{2\gamma a^2}{\pi}\sqrt{1-s^2}$$

Substituting this expression under the integral sign in (3.6), we obtain a formula for determining the size of the area of contact: $\gamma a^2 = P$, which coincides with the known formula.

4. Let us touch upon the problem of rigorous proof of the formal solution (2.5) of the equation (0.1). We can see that fulfilling e.g. the conditions: a) systems of elements (2.3) are B-linearly independent; b) the operator L is defined and continuous everywhere in H'; c) the first series of (1.1) converges in H to the element f, and d) series (2.5) converges in H, is sufficient for a rigorous proof of validity of the solution obtained. The condition a) can be conveniently verified with help of the following theorem:

T h e o r e m 2. If the system $\{\varphi_n\}$ is linearly independent and the system $\{\psi_n\}$ is defined by the formula

$$\psi_n = K\varphi_n, \; n = 0, 1, 2, \ldots \tag{4.1}$$

where the following inequality holds for the operator K (acting from H' into H) in its domain of definition:

$$(f,\ Kf) > 0, \quad f \neq 0 \tag{4.2}$$

(i.e. the operator K is positive [6]), then the systems $\{\varphi_n\}$ and $\{\psi_n)$ are B-linearly independent.

To prove this we substitute into (4.2) an arbitrary element of the form

$$f = \sum_{j=0}^{n} c_j \varphi_j \quad \left(\sum_{j=0}^{n} c_j^2 \neq 0\right) \tag{4.3}$$

and use (4.1) to obtain

$$(f, Kf) = \sum_{j=0}^{n} \sum_{k=0}^{n} (\psi_j, \varphi_k)\, c_j c_k > 0 \tag{4.4}$$

Strict inequality is guaranteed here for any element of the form (4.3), since the case $f = 0$ is exluded by virtue of the condition that the system $\{\varphi_n\}$ is linearly independent. The inequality (4.4) on the other hand implies that the B-matrix generates a positive quadratic form, and this as we know from [12] implies the strict positiveness of its subsequent principal minors which in the present case coincide with the B-determinants, and this proves the theorem.

In accordance with this theorem the condition a) will hold for the case (2.13) provided that the equation $L^* g = 0$ has a null solution only, since we then have $\psi_n = LL^* \varphi_n$ and therefore

$$(f,\ LL^* f) = (L^* f,\ L^* f) > 0,\ L^* f \neq 0\ (f \neq 0) \tag{4.5}$$

In this case a Fredholm operator, say, will be fully acceptable as L, i.e.

$$L = I - \lambda K \tag{4.6}$$

(K is a completely continuous operator) and the values of λ coinciding with the eigenvalues of the operator L can be eliminated.

In the case (2.10) the condition a) will hold by virtue of the theorem, if the operator L itself is positive, i.e. if the inequality (4.2) holds for this operator. Such a situation may occur e.g. when the operator is integral and generated by a kernel of the form

$$L(x, y) = \int_{\Omega} \rho(t)\, l(x, t)\, l(y, t)\, d\Omega \tag{4.7}$$

$$\rho(t) \geqslant 0 \quad (x, y, t \in \Omega)$$

Next we explain when the condition c) holds. We restrict ourselves to the cases (2.10) and (2.13). Equation (4.1) holds in the first case only when $K = L$, and in the second case only when $K = LL^*$, i.e. the operators K are symmetric in both cases and have the property (4.2). It can be verified that for such operators the choice of the complete system $\{\varphi_n\}$ implies the completeness of the system $\{\psi_n\}$. Moreover, the B-systems (1.4) generated by the systems $\{\varphi_n\}$ and $\{\psi_n\}$ will be complete in H.

Further we note that the symmetric operator defined everywhere in H is bounded [6], and this means that the system $\{\xi_n\}$ is a Bessel system by virtue of the known assertion (see [2], p. 438). This is due to the fact that K is a bounded positive symmetric operator which transforms the system $\{\xi_n\}$, according to (2.11) and (2.14), into its conjugate $\{\eta_n\}$.

It is known (see [13]), that the symmetric operators with property (4.2) have inverse operators which are also symmetric. The latter will be defined everywhere in H provided that L is a Fredholm operator, i.e. provided that it can be written in the form (4.6). Indeed, in the case (2.13) when $K^{-1} = (LL^*)^{-1} = L^{*-1}L^{-1}$, this follows from the fact that when λ is not an eigenvalue, equations $Lg = f$ and $L^*g = f$ have a unique solution for every $f \in H$. In the case (2.10), when $K^{-1} = L^{-1}$, the assumption that L is positive made in order to satisfy the condition a), implies that it is symmetric (see [14], p. 352) and has positive eigenvalues. Consequently in the present case K^{-1} is defined everywhere in H provided that $\lambda < 0$ in (4.6), since the equation $Lg = f$ has a unique solution for every $f \in H$.

Thus K^{-1} is a bounded symmetric operator in both cases (see [6]) and the relation $\xi_n = K^{-1}\eta_n$ holds by virtue of (2.11) and (2.14). This implies that $\{\xi_n\}$ is not only a Bessel system, but also a Hilbert system (see [2], p. 439), and this means that $\{\xi_n\}$ and $\{\eta_n\}$ are Riesz-Fischer systems (see [2], p. 440). The latter fact implies the existence of constants M and N such, that for every $f \in H$

$$\sum_{m=0}^{\infty} (f, \xi_m)^2 \leqslant M^2 \|f\|^2, \quad \sum_{m=0}^{\infty} (f, \eta_m)^2 \leqslant N^2 \|f\|^2 \tag{4.8}$$

The above inequalities make possible the proof of weak convergence of the series (1.1) in H, and this is equivalent to their convergence in H to f (see [6]). Indeed, when the first series of (1.1) converges e.g. weakly to f, then by virtue of the weak completeness [6] of the Hilbert space H it is sufficient to show that the sequences $(f^{(n)}, g)$ $(n = 0, 1, 2, \ldots)$, where g denotes any element of H and $f^{(n)}$ are the partial sums of the series of (1.1), are bounded. This is easily done with help of the inequalities (4.8).

Next we find whether the last condition d) holds. In the case (2.10) the convergence of the series (2.5), or more accurately of (2.12), follows from the fact that the equations

$$(L = L^*) \quad (f, \xi_m) = (Lg, \xi_m) = (g, L\xi_m) = (g, \eta_m)$$

transform it into one of the biorthogonal series (1.1) the convergence of which was proved for any element belonging to H. On the other hand, the convergence of (2.5) in the case (2.13) can be proved provided that we assume that the system $\{\xi_n\}$ is, in the present case, orthonormal and the inequalities (4.8) hold.

Thus the formal solution (2.5) of (0.1) is shown to hold in the case (2.13), provided that L is a Fredholm operator, i.e. that it can be written in the form (4.6), while in the case (2.10) it must also be positive. To ensure the latter property, it is sufficient (see [14], p. 352) to assume that $\lambda < 0$ and K in (4.6) are self-conjugate.

A strict proof of (2.5) demands, in the general case, the foreknowledge of finer properties of the B-systems.

REFERENCES

1. P o p o v, G. Ia. Exact solution of the problem of oscillation of a plate clamped along its contour. Dokl. Akad. Nauk SSSR, Vol. 233, No. 5, 1977.

2. K a c z m a r z, S. and S t e i n h a u s, H. Theorie der Orthogonalreihen. N. Y. Chelsea Publishing Co. 1951.

3. B a t e m a n, H. and E r d e l y i, A. Higher Transcendental Functions (Bessel functions, cylindrical parabolic functions, orthogonal polynomials). N. Y. McGraw–Hill, 1953–55.

4. G o k h b e r g, I. Ts. and K r e i n, M. G. Theory and applications of Volterra operators in Hilbert space. Providence American Mathematical Society, 1970.

5. K o z i n, I. V. Method of solving the equation $f = Au$. Tr. Leningr. Univ. aviats. priborostroeniia, No. 54, 1967.

6. A k h i e z e r, N. I. and G l a z m a n, I. M. Theory of Linear Operators in Hilbert Space. Moscow, "Nauka", 1966.

7. M i k h l i n, S. G. Variational Methods in Mathematical Physics. (English translation), Pergamon Press, Book No. 10146, 1964.

8. T r i c o m i, F. G. Integral Equations. N. Y. Interscience, 1957.

9. S h t a e r m a n, I. Ia. Contact Problem of the Theory of Elasticity. Moscow–Leningrad, Gostekhizdat, 1949.

10. P o p o v, G. Ia. On an extraordinary property of the Jacobi polynomials. Ukr. matem. Zh. Vol. 20, No. 4, 1968.

11. G r a d s h t e i n, I. S. and R i z h i k, I. M. Tables of Integrals, Sums, Series and Products. Moscow, Fizmatgiz, 1962. (see also English translation, Pergamon Press, Book No. 09832, 1963).

12. G a n t m a k h e r, F. R. and K r e i n, M. G. Oscillatory Matrixes and Kernels, and Small Oscillations of Mechanical Systems. Moscow–Leningrad, Gostekhizat, 1950.

13. V u l i k h, B. Z. Introduction to Functional Analysis. Pergamon Press, Addison Wesley, Publ. Co., U. S. A. Distributors, 1963.

14. R u d i n, U. Functional Analysis. Moscow, "Mir", 1975.

Translated by L. K.